An elliptic problem involving large advection

A. Aghajani[*] C. Cowan[†]

August 13, 2021

Abstract

We consider positive classical solutions of

$$\begin{cases} -\Delta u + \lambda a(x) \cdot \nabla u = u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $p > 1$, a is a smooth divergence free vector field and $\lambda > 0$ is a large parameter. Under certain assumptions on $a(x)$ and (or) assumptions on the existence of first integrals of $a(x)$ we show there is a subsequence of smooth positive solutions which converge to a nonzero first integral of $a(x)$ as $\lambda \to \infty$.

Key words: Second-order elliptic equations; Drift term; Divergence free.

2010 Mathematics Subject Classification: 35J15, 35B45.

1 Introduction

In this work we are interested in examining positive classical solutions of

$$(Q)_\lambda \qquad \begin{cases} -\Delta u + \lambda a(x) \cdot \nabla u = u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases}$$

where $\lambda > 0$ is a parameter and where Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 2$) with smooth boundary. We will assume that a is a smooth and divergence free vector field. Our main interest will be the asymptotics of positive solutions as $\lambda \to \infty$.

[*]School of Mathematics, Iran University of Science and Technology, Narmak, Tehran, Iran. Email: aghajani@iust.ac.ir.

[†]Department of Mathematics, University of Manitoba, Winnipeg, Manitoba, Canada R3T 2N2. Email: craig.cowan@umanitoba.ca. Research supported in part by NSERC.

1.1 Background

We begin by considering the case of $(Q)_\lambda$ when $a = 0$. In this case the equation becomes

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

For $N \geq 3$ we define the critical exponent $p_s = \frac{N+2}{N-2}$ and note that it is related to the critical Sobolev imbedding exponent $2^* := \frac{2N}{N-2} = p_s + 1$. For $1 < p < p_s$ one has that $H_0^1(\Omega)$ is compactly imbedded in $L^{p+1}(\Omega)$ and hence one can show the existence of a positive minimizer of

$$\min_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\left(\int_\Omega |u|^{p+1} dx\right)^{\frac{2}{p+1}}}.$$

This positive minimizer is a positive solution of (1), see for instance the book [39]. Note in the case of $N = 2$ we have the existence of a positive minimizer for any $p > 1$. For $p \geq p_s$, $H_0^1(\Omega)$ is no longer compactly imbedded in $L^{p+1}(\Omega)$ and so to find positive solutions of (1) one needs to take other approaches. For $p \geq p_s$ the well known Pohozaev identity [31] shows there are no positive solutions of (1) provided Ω is star shaped. For general domains in the critical/supercritical case, $p \geq p_s$, the existence versus nonexistence of positive solutions of (1) is a very delicate question; see [1, 8, 17, 16, 15, 14, 18, 30, 27, 28, 36, 37].

The various critical exponents. In this work a few well known critical exponents make an appearance; $p_{BT} := \frac{N+1}{N-1} < \frac{N}{N-2} < p_s$ at least when $N \geq 3$. To give a background we introduce the equation

$$\begin{cases} -\Delta u = b(x) u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where Ω a bounded domain with smooth boundary in $\mathbb{R}^N$ $(N \geq 3)$ and b is a smooth positive function. For $b(x) = b_0 > 0$ and Ω star shaped we know there is no positive classical solution for any $p \geq p_s$ as mentioned above.

For general b, in 1977, Brezis and Turner [6] showed that an $H_0^1(\Omega)$ solution is smooth provided $1 < p < p_{BT}$ the Brezis-Turner exponent. For b constant this was improved to the full range $1 < p < p_s$ in [22, 13]. In [33] it was shown that *very weak* solutions of (2) are smooth provided $1 < p < p_{BT}$. Here we define a *very weak* solution of (2) by $u \in L^1(\Omega)$ (with some minor other assumptions) such that

$$\int_\Omega (-\Delta \phi) u dx = \int_\Omega b(x) u^p \phi dx, \quad \forall \phi \in \{\phi \in C^2(\overline{\Omega}) : \phi = 0 \text{ on } \partial\Omega\},$$

note this idea of a very weak solution is essentially the same as a distributional solution but the boundary condition for u has been encoded into the definition by slightly enlarging the class of test functions.

If one sets $u(x) = |x|^{-\alpha} - 1$ with $\alpha = \frac{2}{p-1}$ there is some $C_p > 0$ such that u is a very weak solution of $-\Delta u = C_p(u+1)^p$ in B_1 with $u = 0$ on ∂B_1 provided $\frac{N}{N-2} < p < p_s$. This leaves open the question of whether very weak solutions of (2) are smooth for $p_{BT} < p < \frac{N}{N-2}$. In [38] it was shown that for all $p > p_{BT}$ there is some bounded positive b and a very weak solution of (2) which is neither bounded or $H_0^1(\Omega)$. This shows the optimality of the Brezis-Turner exponent.

Adding the advection term. We now return to $(Q)_\lambda$. For the case of general $a(x)$ the equation is no longer variational and hence one cannot find critical points of a suitable energy to find positive solutions of $(Q)_\lambda$. We point out the special case of $a(x) = \nabla\gamma(x)$ where γ is a scalar function; in this case the equation is variational and hence one can find critical points of a suitable energy. In particular a minimizer of

$$E_\lambda(u) := \frac{\int_\Omega e^{-\lambda\gamma}|\nabla u|^2 dx}{\left(\int_\Omega e^{-\lambda\gamma}|u|^{p+1}dx\right)^{\frac{2}{p+1}}}$$

satisfies $(Q)_\lambda$.

We now consider the idea of large drift problems. We first mention that these large drift problems have attracted a lot of attention in the context of travelling fronts, see for instance [24, 25, 26, 3, 4, 35, 40] and also [5]. Of key importance to large drift problems is the notion of a first integral which we now define.

Definition 1. *(First integrals) We say $\psi \in H_0^1(\Omega)$ is a first integral for a provided $\psi \neq 0$ and $a(x)\cdot\nabla\psi(x) = 0$ for a.e. $x \in \Omega$. We denote the class of first integrals of a by $\mathcal{A}$. Denote $\mathcal{A}_+ := \{\psi \in \mathcal{A} : \psi \geq 0 \text{ a.e.}\}$.*

We mention here a result regarding a linear problem with large advection. Berestycki, Hamel and Nadirashvili in [3] examined the eigenvalue problem given by

$$\begin{cases} -\Delta\phi + ta(x)\cdot\nabla\phi = \mu_t\phi & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where μ_t denotes the principal eigenvalue and their interest is what happens when $t \to \infty$. For smooth divergence free vector fields a they showed that

$\{\mu_t\}_t$ is bounded if and only if a has a first integral in $H_0^1(\Omega)$. Moreover, if a has a first integral in $H_0^1(\Omega)$ then

$$\mu_t \to \min_{w\in\mathcal{A}} \frac{\int_\Omega |\nabla w|^2}{\int_\Omega w^2} \quad as\ t\to\infty, \tag{4}$$

and the minimum in the right-hand side of (4) is achieved.

We let $\delta(x) := dist(x,\partial\Omega)$ denote the Euclidean distance from x to $\partial\Omega$. For $\varepsilon>0$ small set $\Omega_\varepsilon := \{x\in\Omega : \delta(x)<\varepsilon\}$.

Assumptions on $a(x)$. There are two assumptions we impose on $a(x)$ that allow us to obtain results. The important issue is whether $a(x)$ has a first integral with suitable properties. We note this can be quite a complex issue which we don't want to address here, the interested reader should see [24, 25, 26]).

(A1) a is smooth, divergence free and compactly supported in Ω. In this case one can easily construct a smooth positive first integral ψ with $\Delta\psi\ge 0$ near $\partial\Omega$ (see below how to get this condition on $\Delta\psi$) and $\psi\ge c\delta$ in Ω for some $c>0$.

(A2) a is smooth, divergence free and there exists some smooth $\psi\in\mathcal{A}_+$ and $c_1>0$ such that $\psi(x)\ge c_1\delta(x)$ in Ω. Note under this assumption that $|\nabla\psi|$ is bounded away from zero near $\partial\Omega$. By replacing ψ with $x\mapsto m(\psi(x))^2+\psi(x)$ (for large m) we can assume $\Delta\psi\ge 0$ near $\partial\Omega$. So in case (A2) we have the existence of a smooth positive first integral ψ with $\Delta\psi\ge 0$ near $\partial\Omega$ and $\psi\ge c\delta$ in Ω for some $c>0$.

Note in both the above cases we can assume there is some $C_2>0$ such that $\psi\le C_2\delta$ in Ω; this follows from the smoothness of ψ. So in both cases we will assume ψ is comparable to δ. Note also that in (A1) the only real assumption is on a. In the case of (A2) we are making an assumption on the existence of positive smooth first integral of a.

Acknowledgments. We would like to thank the anonymous referee for the detailed report which greatly increased the readability of the work.

1.2 Main results

Proposition 1. *(Existence of positive smooth solution) Assume* $1 < p < \frac{N+2}{N-2}$ *and a is smooth and bounded. Then there is a smooth positive solution of* $(Q)_\lambda$*. If we further assume divergence of a is zero then there is some* $C_1 > 0$ *(independent of* λ*) such that any positive solution of* $(Q)_\lambda$ *satisfies* $\sup_\Omega u \geq C_1$*.*

We now state our mains results in the case of the asymptotics in λ.

Theorem 1. *Let a satisfy* $(A1)$*,* ψ *is as in condition* $(A1)$ *and* $1 < p < \frac{N}{N-2}$*. Let* u_m *denote a sequence of smooth solutions of* $(Q)_{\lambda_m}$ *with* $\lambda_m \to \infty$*. Then there is some* $C_i > 0$ *such that* $0 < C_1 \leq \sup_\Omega u_m \leq C_2$*. Moreover we have*

$$\int_\Omega (a(x) \cdot \nabla u_m(x))^2 dx \leq \frac{C}{\lambda_m} \to 0.$$

Also there is some $u \in H_0^1 \backslash \{0\} \cap \mathcal{A}_+$ *and some* $\{u_{m_k}\}_k$ *such that* $u_{m_k} \rightharpoonup u$ *in* H_0^1 *and* $u_{m_k} \to u$ *in* L^T *for all* $T < \infty$*.*

Theorem 2. *Let a satisfy* $(A2)$ *and let* $1 < p < \frac{N+1}{N-1}$*. Let* u_m *denote a sequence of smooth positive solutions of* $(Q)_{\lambda_m}$ *with* $\lambda_m \to \infty$*. Then there is some* $C_i > 0$ *such that* $0 < C_1 \leq \sup_\Omega u_m \leq C_2$*. Moreover we have*

$$\int_\Omega (a(x) \cdot \nabla u_m(x))^2 dx \leq \frac{C}{\lambda_m} \to 0.$$

Also there is some $u \in H_0^1 \backslash \{0\} \cap \mathcal{A}_+$ *and some* $\{u_{m_k}\}_k$ *such that* $u_{m_k} \rightharpoonup u$ *in* H_0^1 *and* $u_{m_k} \to u$ *in* L^T *for all* $T < \infty$*.*

Some open problems. It would be interesting to investigate whether these results should hold for a larger range of p. For instance can one extend Theorem 1 and (or) 2 to all $1 < p < p_s$? Can one extend Theorem 2 to all $1 < p < \frac{N}{N-2}$? Since one is dealing with smooth solutions the most natural approach is to try a rescaling/blow up approach; of course $\lambda_m \to \infty$ adds major difficulties to this approach and this is why we took a more integral estimate approach so the advection term drops out.

2 Proofs

2.1 Existence results for fixed λ

We begin with some lemmas.

Lemma 1. *Let b be a divergence free smooth vector field and v a smooth positive solution of $-\Delta v + b(x) \cdot \nabla v = v^p$ in Ω with $v = 0$ on $\partial\Omega$. Let*

$$L(\phi) := -\Delta\phi + b(x) \cdot \nabla\phi - pv(x)^{p-1}\phi.$$

Then $\lambda_1(L) < 0$ (here $\lambda_1(L)$ is the first eigenvalue of L in $H_0^1(\Omega)$).

Proof. Let L^* denote the adjoint of L, ie. $L^*(\psi) = -\Delta\psi - b \cdot \nabla\psi - pv^{p-1}\psi$. Note that $\lambda_1(L) = \lambda_1(L^*)$. Let $\psi > 0$ denote the first eigenfunction of L^* in $H_0^1(\Omega)$, ie.

$$L^*(\psi) = \lambda_1(L^*)\psi = \lambda_1(L)\psi \quad \text{in } \Omega, \quad \psi = 0 \quad \text{on } \partial\Omega.$$

Multiply the equation for ψ by v and integrate to obtain

$$\begin{aligned}\lambda_1(L)\int_\Omega \psi v &= \int_\Omega L^*(\psi)v \\ &= \int_\Omega (-\Delta v + b \cdot \nabla v)\psi - p\int_\Omega v^p\psi \\ &= (1-p)\int_\Omega v^p\psi\end{aligned}$$

and since $v, \psi > 0$ and $p > 1$ we see that $\lambda_1(L) < 0$. □

Lemma 2. *Let b be divergence free smooth vector field and $v > 0$ denote a smooth solution of $-\Delta v + b \cdot \nabla v = v^p$ in Ω with $v = 0$ on $\partial\Omega$. Set $L_1(\zeta) := -\Delta\zeta - b \cdot \nabla\zeta$. Then one has the estimate*

$$p\|v\|_{L^\infty}^{p-1} > \lambda_1(L_1) \geq \lambda_1(-\Delta),$$

where this last term is just the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$.

Proof. Let $\phi, \psi > 0$ denote the first eigenfunctions (with zero Dirichlet boundary conditions);

$$\begin{aligned}-\Delta\phi + b \cdot \nabla\phi &= pv^{p-1}\phi + \mu\phi \quad \text{in } \Omega, \\ -\Delta\psi - b \cdot \nabla\psi &= \lambda_1(L_1)\psi \quad \text{in } \Omega,\end{aligned}$$

and note that $\mu < 0$ from the previous lemma. Multiply the first equation by ψ and integrate to see that

$$\begin{aligned}\mu\int_\Omega \phi\psi + \int_\Omega pv^{p-1}\phi\psi &= \int_\Omega L_1^*(\phi)\psi \\ &= \int_\Omega \phi L_1(\psi) \\ &= \int_\Omega \phi\lambda_1(L_1)\psi.\end{aligned}$$

But since $\mu < 0$ we have

$$\int_\Omega p v^{p-1} \phi \psi \geq \int_\Omega \lambda_1(L_1) \phi \psi,$$

and then the desired result follows since $\phi, \psi > 0$. To complete the proof we show that $\lambda_1(L_1) \geq \lambda_1(-\Delta)$. Multiply the equation for ψ and integrate to arrive at

$$\begin{aligned} \lambda_1(L_1) \int_\Omega \psi^2 dx &= \int_\Omega |\nabla \psi|^2 dx - \int_\Omega (b \cdot \nabla \psi) \psi dx \\ &= \int_\Omega |\nabla \psi|^2 dx \quad \text{since } div(b) = 0 \\ &\geq \lambda_1(-\Delta) \int_\Omega \psi^2 dx \quad \text{minimality of } \lambda_1(-\Delta), \end{aligned}$$

and this completes the proof. □

We show that for $1 < p < \frac{N+2}{N-2}$ and $\lambda > 0$ there exists a positive classical solution of $(Q)_\lambda$. To do this we use essentially the exact degree theory argument from [34] where they examined positive solutions of higher order equations with lower order terms. For more details on the degree theory argument one can see the book [2]. In addition there is some $C_1 > 0$ (independent of λ) such that any positive classical solution of $(Q)_\lambda$ satisfies $\sup_\Omega u \geq C_1$.

Proof of Proposition 1. Set $\lambda = 1$ for simplicity and define $K_t : L^\infty(\Omega) \to L^\infty(\Omega)$ via $K_t(u) = v$ where v satisfies

$$-\Delta v + a(x) \cdot \nabla v = |u|^p + t \qquad \Omega, \qquad v = 0 \quad \partial\Omega.$$

We first note that K_t is a compact operator for each t. We first show that for large enough t that there is no solution of $u - K_t(u) = 0$. Let $L(\phi) := -\Delta \phi + a(x) \cdot \nabla \phi$ and let L^* denote its formal adjoint with first eigenpair given by (λ_1^*, ψ) where $\psi > 0$. Multiply the equation by ψ and integrate by parts to see

$$\int_\Omega \left(u^p + t - \lambda_1^* u\right) \psi(x) dx = 0$$

and hence we have

$$t \leq \sup_{0 \leq u} \left(\lambda_1^* u - u^p\right),$$

which gives us an upper bound on t. Let T be large enough such that there is no solution of $u - K_T(u) = 0$. Let B_R denote the open ball of radius R centered at the origin in $L^\infty(\Omega)$. Note we can write this nonexistence result as $0 \notin (I - K_T)(B_R)$ and this implies $deg(I - K_T, B_R, 0) = 0$ for all $R > 0$; see [2] or Theorem 2.1 in [29] for some well known results regarding Leray-Schauder degree theory.

Now we claim that for R large enough that we have $0 = deg(I - K_T, B_R, 0) = deg(I - K_0, B_R, 0)$. We prove the result via the homotopy invariance, see Theorem 2.1 (iii) in [29]. If we assume the result is false then there is some $R_m \to \infty$ and $0 < u_m \in \partial B_{R_m}$ (and $0 \le t_m \le T$) such that $u_m - K_{t_m}(u_m) = 0$. So u_m satisfies

$$-\Delta u_m + a(x) \cdot \nabla u_m(x) = u_m^p + t_m \quad \Omega, \qquad u_m = 0 \quad \partial\Omega$$

where $x_m \in \Omega$ satisfies $u_m(x_m) = \|u_m\|_{L^\infty} = R_m \to \infty$.

Define $v_m(x) = \frac{u_m(x_m + r_m x)}{R_m}$ for $x \in \Omega_m := \{x \in \mathbb{R}^N : x_m + r_m x \in \Omega\}$. Then v_m satisfies $0 \le v_m \le 1$ with $v_m(0) = 1$ and

$$-\Delta v_m(x) + b_m(x) \cdot \nabla v_m(x) = r_m^2 R_m^{p-1} v_m(x)^p + \frac{t_m r_m^2}{R_m} \qquad \text{for } x \in \Omega_m,$$

with $v_m = 0$ on $\partial\Omega_m$ where $b_m(x) := r_m a_m(x_m + r_m x)$. Now pick r_m such that $r_m^2 R_m^{p-1} = 1$ and so $r_m \to 0$. Define $\delta_m := dist(x_m, \partial\Omega)$ and we need to consider a few cases. After passing to subsequences we can assume one of the following holds:

case 1. $\frac{r_m}{\delta_m} \to 0$,
case 2. $\frac{r_m}{\delta_m} \to \gamma \in (0, \infty)$,
case 3. $\frac{r_m}{\delta_m} \to \infty$.

Case 1. In this case $\Omega_m \to \mathbb{R}^N$ and after some compactness arguments we have (for a subsequence) $v_m \to v$ in $C^{1,\alpha}_{loc}(\mathbb{R}^N)$ and v solves: $0 \le v \le 1$ with $v(0) = 1$ and

$$-\Delta v = v^p \qquad \text{in } \mathbb{R}^N. \tag{5}$$

Since $1 < p < \frac{N+2}{N-2}$ this contradicts well known results of [22, 23], at least in the case of $N \ge 3$. In the case of $N = 2$ the proof of the needed Liouville theorem has a simple proof which we include for the sake of the reader. For

$R > 1$ we let $0 \leq \phi_R \in C_c^\infty(B_{2R})$ with $\phi_R = 1$ in B_R and $0 \leq \phi_R \leq 1$. Moreoever there is some $C > 0$ such that $|\nabla \phi_R| \leq CR^{-1}, |\Delta \phi_R| \leq CR^{-2}$ for all $R > 1$. Multiply (5) by ϕ_R^k (k large to be picked later) and integrate by parts to arrive at

$$\begin{aligned} \int u^p \phi^k dx &= \int u(-\Delta \phi^k) dx \\ &= \int u(-k)(k-1)\phi^{k-2}|\nabla \phi|^2 dx + \int uk\phi^{k-1}(-\Delta\phi) dx \\ &\leq \int uk\phi^{k-1}(-\Delta \phi) dx \\ &\leq \left(\int u^p \phi^{(k-1)p} dx \right)^{\frac{1}{p}} k \left(\int |\Delta \phi|^{p'} dx \right)^{\frac{1}{p'}}, \end{aligned}$$

where p' is the conjugate of p and note for large k we have $(k-1)p \geq k$ and hence $\phi^k \geq \phi^{(k-1)p}$. Using this we can combine the integrals to arrive at

$$\int_{B_R} u^p dx \leq CR^{N-2p'},$$

and recalling $N = 2$ we see that $N - 2p' = 2 - 2p' < 0$ and hence sending $R \to \infty$ we see $\int_{\mathbb{R}^N} u^p dx = 0$ which shows $u = 0$. Note in higher dimensions this approach proves a Liouville theorem for super solutions of $-\Delta u = u^p$ in $\mathbb{R}^N$ for $1 < p < \frac{N}{N-2}$.

Case 2. After a suitable rotation of co-ordinates one can see that $\Omega_m \to H := \{x \in \mathbb{R}^N : x_N > \frac{-1}{\gamma}\}$. By a limiting argument there is some $0 \leq v \leq 1$ such that $v(0) = 1$, $0 \leq v \leq 1$ with $-\Delta v = v^p$ in H with $v = 0$ on ∂H. This contradicts some Liouville theorems on the half space, for instance, in [10], it was shown the Liouville theorem holds for $1 < p < \frac{N+1}{N-3}$ (the critical exponent in dimension $N - 1$). This completes the proof of this case. For the interest of the reader we will give a bit more details on these half space Liouville theorems since they have had some interest in the last number of years. Lets now assume the half space is $\mathbb{R}^N_+ = \{x \in \mathbb{R}^N : x_N > 0\}$. In [10] the idea was to use the Moving Plane Method to show a positive solution was increasing in x_N and then he considered

$$w(x') := \lim_{x_N \to \infty} v(x', x_N)$$

which can be shown to satisfy

$$-\Delta_{\mathbb{R}^{N-1}} w(x') = w(x')^p \qquad \text{in } \mathbb{R}^{N-1},$$

with w bounded and positive. He then applied known results from [22, 23], in the case of $N-1 \geq 3$, and similar results in lower dimensions, to get the desired contradiction. The next major improvement was in [21]. The interest here was in Liouville theorems for stable (possibly) sign changing solutions of $-\Delta u = |u|^{p-1}u$ in $\mathbb{R}^N$. Here a Liouville theorem was proven for all $1 < p < p_{JL}$, the Joseph-Lundgren exponent. Given a positive solution on the half space which is increasing in x_N one extends oddly to the full space and this solution can be shown to be stable and hence the results of [21] can be applied. This substantially increased the range of p from the previous result. One should also see [11, 12] for related works.

For bounded solutions on the half space (which is sufficient for our purposes) this Liouville theorem was extended to all $p > 1$ in [7]. This result was recently improved by removing the boundedness assumption of the solution, see [20].

Case 3. Here we are assuming $\frac{r_m}{\delta_m} \to \infty$; recall $r_m^2 R_m^{p-1} = 1$. So note we have $R_m^{\frac{p-1}{2}} \delta_m \to 0$. Define $v_m(x) := \frac{u_m(x_m + \delta_m x)}{R_m}$ for $x \in \Omega_m := \{x \in \mathbb{R}^N : x_m + \delta_m x \in \Omega\}$. Then note that v_m satisfies

$$-\Delta v_m(x) + b_m(x) \cdot \nabla v_m(x) = R_m^{p-1} \delta_m^2 v_m(x)^p + \frac{t_m \delta_m^2}{R_m} \quad \text{for } x \in \Omega_m \quad (6)$$

with $v_m = 0$ on $\partial\Omega_m$; where $b_m(x) := \delta_m a(x_m + \delta_m x)$. Also note that $0 \leq v_m \leq 1$ with $v_m(0) = 1$. Also note that (after suitable rotation of domain) we have $\Omega_m \to \{x \in \mathbb{R}^N : x_N > -1\} =: H$. Also note that

$$\sup_{\Omega_m} \left(R_m^{p-1} \delta_m^2 v_m(x)^p + \frac{t_m \delta_m^2}{R_m} \right) \to 0$$

as $m \to \infty$. So from this we can pass to a limit to find some v such that $-\Delta v = 0$ in H with $v = 0$ on ∂H with $v(0) = 1$ and $0 \leq v \leq 1$. But this contradicts the strong maximum principle for harmonic functions. Combining these three cases proves the claim. For more detailed calculations direct the reader to [19].

We now show that for large enough $R > 0$ and small enough $\varepsilon > 0$ we have

$$0 = deg(I - K_0, B_R, 0) = deg(I - K_0, B_R \backslash B_\varepsilon, 0) + deg(I - K_0, B_\varepsilon, 0). \quad (7)$$

The first equality we already have and hence to prove the result we need to verify the second equality. To prove the result we will use the additivity property of Leray-Schauder degree theory; see Theorem 2.1 part (i) [29]. First note we can write $B_R = B_\varepsilon \cup \left(B_R \backslash \overline{B_\varepsilon}\right)$. Provided we can show that for small enough $\varepsilon > 0$ that $u - K_0(u) \neq 0$ on ∂B_ε then we can apply the additivity result and hence (7) holds. So we need to show there is no solution of $(Q)_1$ with $\|u\|_{L^\infty} = \varepsilon$; but this follows from Lemma 2.

We now show that $deg(I - K_0, B_\varepsilon, 0) = 1$ and hence $deg(I - K_0, B_R \backslash \overline{B_\varepsilon}, 0) = -1$ and we have our desired nonzero solution.

Consider the homotopy $u - tK_0(u)$ and we claim that for $0 < \varepsilon$ small enough we have $1 = deg(I, B_\varepsilon, 0) = deg(I - K_0, B_\varepsilon, 0)$.

Suppose not, so there is $\varepsilon_m \searrow 0$ and $\|u_m\|_{L^\infty} = \varepsilon_m$ such that $u_m - t_m K_0(u_m) = 0$ ($t_m \in [0,1]$). So we have $u_m > 0$ satisfies $-\Delta u_m + a \cdot \nabla u_m = t_m u_m^p$ in Ω with $u_m = 0$ on $\partial\Omega$ and the maximum principle shows we must have $t_m > 0$. Set $v_m := \frac{u_m}{\varepsilon_m}$ and note that $0 < v_m$ satisfies $\sup_\Omega v_m = 1$ and

$$-\Delta v_m + a \cdot \nabla v_m = t_m \varepsilon_m^{p-1} v_m^p \quad \Omega, \quad v_m = 0 \quad \partial\Omega.$$

Now note the right hand side of the equation converges to zero in $L^\infty(\Omega)$ and hence we must have $v_m \to 0$ in $C^1(\overline{\Omega})$, contradicting the fact that v_m is normalized in L^∞.

We now prove the desired lower bound on $\sup_\Omega u$; which essentially follows from the above argument. Suppose u_m a positive smooth solution of $(Q)_{\lambda_m}$ and we suppose $\varepsilon_m := \sup_\Omega u_m \to 0$. Set $v_m := \frac{u_m}{\varepsilon_m}$ and then $\sup_\Omega v_m = 1$ and one has $-\Delta v_m + \lambda_m a(x) \cdot \nabla v_m = \varepsilon_m^{p-1} v_m^p$ in Ω with $v_m = 0$ on $\partial\Omega$. Then using Theorem A (see below) we see that $\|v_m\|_{L^\infty} \to 0$; a contradiction. □

2.2 Asymptotics in λ; estimates on solutions of $(Q)_\lambda$

We begin with a theorem from [5].

Theorem A. *([5]) Suppose*

$$-\Delta u + b(x) \cdot \nabla u = f(x) \ \text{in } \Omega, \qquad u = 0 \ \text{on } \partial\Omega,$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$. For all $p > \frac{N}{2}$ there is some $C_p > 0$ such that for all sufficiently regular vector fields b with $div(b) = 0$ one has $\|u\|_{L^\infty} \leq C_p \|f\|_{L^p}$ for $p > \frac{N}{2}$. (The important point here is that C_p is independent of b). We give a sketch of a proof of this result in the Appendix.

Remark 1. *The proof of the above theorem was proven using some parabolic methods. One can use a more standard elliptic proof; either a De Giorgi method or a Moser iteration type argument to prove the result. Using this approach one sees they can weaken the divergence free condition to $div(b) \leq 0$.*

Lemma 3. (*Initial integral bound*). *Let a satisfy $(A1)$ or $(A2)$ and suppose ψ is a positive smooth first integral of $a(x)$ promised in $(A1)$ and $(A2)$. Then there is some $C = C(p)$ such that for all positive smooth solutions u of $(Q)_\lambda$ we have*

$$\int_\Omega u^p \psi dx \leq C.$$

(*Note the estimate is independent of λ*).

Proof. We assume we are in the case of (A2); in the case of (A1) the proof is similar. Let ψ denote a smooth positive first integral of $a(x)$ which is comparable to the distance function near $\partial\Omega$ and with $\Delta\psi \geq 0$ near $\partial\Omega$; lets say $\Delta\psi \geq 0$ in Ω_ε. Multiplying $(Q)_\lambda$ by ψ and integrating gives

$$\int_\Omega u^p \psi dx = \int_\Omega (-\Delta\psi) u dx \leq \int_{\Omega \setminus \Omega_\varepsilon} \frac{-\Delta\psi}{\psi^{\frac{1}{p}}} \left(u \psi^{\frac{1}{p}} \right) dx$$

and now apply Hölder's inequality on the right to get the desired bound. □

The following result really is just an estimate from [32] for subcritical elliptic problems without boundary conditions.

Proposition 2. *Let a satisfy $(A1)$ (and we assume $a = 0$ in $\Omega_{2\varepsilon}$) and suppose u_m is a smooth positive solution of $(Q)_{\lambda_m}$ where $1 < p < \frac{N+2}{N-2}$. Then there is some $C > 0$ (independent of m) such that*

$$\sup_{\Omega_\varepsilon} u_m \leq C.$$

Proof. Note that $-\Delta u_m = u_m^p$ in $\Omega_{2\varepsilon}$ and so we can apply Theorem 3.1 in [32] to see there is some $C > 0$ (independent of m) such that

$$u_m(x) \leq \frac{C}{dist(x, \partial\Omega_{2\varepsilon})^{\frac{2}{p-1}}},$$

for all $x \in \Omega_{2\varepsilon}$. Now suppose that $R_m = \sup_{\Omega_\varepsilon} u_m \to \infty$. Then there is some $x_m \in \Omega_\varepsilon$ with $dist(x_m, \partial\Omega) \to 0$ such that $\sup_{\Omega_\varepsilon} u_m = u_m(x_m) \to \infty$. Now we can apply the standard blow up argument to obtain a contradiction, which we include some details for the readers convenience. First this blow up argument is essentially contained in the proof of Proposition 1. So, as in the previous proof, we define $v_m(x) = \frac{u_m(x_m + r_m x)}{R_m}$ for $x \in \Omega^m := \{x \in \mathbb{R}^N : x_m + r_m x \in \Omega_\varepsilon\}$. Then v_m satisfies $0 \le v_m \le 1$ with $v_m(0) = 1$ and

$$-\Delta v_m(x) = r_m^2 R_m^{p-1} v_m(x)^p \quad \text{ for } x \in \Omega^m.$$

We can write $\partial\Omega^m = \Gamma_m \cup \widehat{\Gamma}_m$ where $\Gamma_m = \{x \in \mathbb{R}^N : x_m + r_m x \in \partial\Omega\}$ and $\widehat{\Gamma}_m = \{x \in \mathbb{R}^N : dist(x_m + r_m x, \partial\Omega) = \varepsilon\}$ and so note that $v_m = 0$ on Γ_m. Set $\delta_m = dist(x_m, \partial\Omega)$ as before and one needs to consider the same three cases: *case 1.* $\frac{r_m}{\delta_m} \to 0$,
case 2. $\frac{r_m}{\delta_m} \to \gamma \in (0, \infty)$,
case 3. $\frac{r_m}{\delta_m} \to \infty$.

In all three cases we get the same limiting equations and hence we get the same desired contradictions. □

Corollary 1. *Suppose a satisfies* (A1) *and* $1 < p < \frac{N+2}{N-2}$. *Suppose* u_m *is a sequence of smooth positive solutions of* $(Q)_{\lambda_m}$. *Then there is some* $C > 0$ (*independent of* m) *such that*

$$\int_\Omega u_m^p dx \le C.$$

Proof. This follows directly from Lemma 3 and Proposition 2.

□

We now prove our main results; Theorem 1 and 2.

Proof of Theorem 1. Let $u_m > 0$ denote a smooth solution of $-\Delta u_m + \lambda_m a \cdot \nabla u_m = u_m^p$ in Ω with $u_m = 0$ on $\partial\Omega$. We first obtain a L^{p+1} bound on u_m. Let $q := 2^*$ and multiply the equation by u_m and integrate to see $C_N \|u_m\|_{L^q}^2 \le \|\nabla u_m\|_{L^2}^2 = \|u_m\|_{L^{p+1}}^{p+1}$ where the first inequality is coming from the critical Sobolev imbedding $C_N \|\phi\|_{L^q}^2 \le \|\nabla \phi\|_{L^2}^2$ for all $\phi \in H_0^1(\Omega)$. Let $0 < \theta < 1$ be such that

$$\frac{1}{p+1} = \frac{\theta}{p} + \frac{1-\theta}{q},$$

and by L^p interpolation we have

$$\|u_m\|_{L^{p+1}} \le \|u_m\|_{L^p}^{\theta} \|u_m\|_{L^q}^{1-\theta},$$

and combining with the previous inequality we obtain

$$\|u_m\|_{L^{p+1}} \leq \|u_m\|_{L^p}^{\theta} \|u_m\|_{L^{p+1}}^{\frac{(p+1)(1-\theta)}{2}} \leq C\|u_m\|_{L^{p+1}}^{\frac{(p+1)(1-\theta)}{2}},$$

where the constant C is independent of m and is promised by Corollary 1. Note we get an L^{p+1} bound on u_m provided $\frac{(p+1)(1-\theta)}{2} < 1$, which holds exactly when $p < \frac{N}{N-2}$.

An iteration. Let $t_1 > \frac{1}{2}$ and for $k \geq 1$ set

$$t_{k+1} = \frac{qt_k}{2} - \frac{p-1}{2}.$$

In the end we will take $t_1 = 1$ which simplifies some of the computations; but we are being more general now since a priori its not clear this doesn't give a better result.

Suppose $t_k > \frac{1}{2}$ and suppose there is some C_k such that $\|u_m\|_{L^{p+2t_k-1}} \leq C_k$ for all m. Then there is some $\widehat{C}_k$ such that $\|u_m\|_{L^{p+2t_{k+1}-1}} \leq \widehat{C}_k$ for all m.

Proof of the inductive step. If one multiplies $(Q)_{\lambda_m}$ by $u_m^{2t_k-1}$ and integrates by parts (one needs to take a bit of care near $\partial\Omega$ if $t_k < 1$) one arrives at

$$(2t_k - 1)\int_\Omega u_m^{2t_k-2}|\nabla u_m|^2 dx = \int_\Omega u_m^{p+2t_k-1} dx,$$

and the left hand side can be rewritten as

$$\frac{(2t_k-1)}{t_k^2}\int_\Omega |\nabla(u_m^{t_k})|^2 dx.$$

By known results one has $u_m^{t_k} \in H_0^1(\Omega)$ since $t_k > \frac{1}{2}$ (see for instance [9]) and one can now apply the critical Sobolev inequality to this term and combine with the above equality to arrive at

$$\left(\int_\Omega u_m^{t_k q} dx\right)^{\frac{2}{q}} \leq \frac{\tilde{C}_N t_k^2}{2t_k - 1}\int_\Omega u_m^{p+2t_k-1} dx,$$

which we can rewrite as

$$\|u_m\|_{L^{t_k q}}^{2t_k} \leq \frac{\tilde{C}_N t_k^2}{2t_k - 1}\|u_m\|_{L^{p+2t_k-1}}^{p+2t_k-1}.$$

Using the definition of t_{k+1} we see that

$$\|u_m\|_{L^{p+2t_{k+1}-1}}^{2t_k} \leq \frac{\tilde{C}_N t_k^2}{2t_k - 1} \|u_m\|_{L^{p+2t_k-1}}^{p+2t_k-1},$$

which completes the proof of the inductive step.

We now examine the cobweb diagram for t_k and, for the time being, we omit the needed assumption that $t_k > \frac{1}{2}$. If $t_1 > \frac{p-1}{q-2}$ the diagram shows that $t_k \to \infty$. Noting that when one take $t_k = \frac{1}{2}$ we have $p + 2t_k - 1 = p$ one might suspect that they can improve on the range of p we have given; we are unable to make any improvements like this.

If we now return to our case with $t_1 = 1$ we see that $t_k \to \infty$ and $t_k > 1$ for all $k \geq 2$. Fix $\frac{N}{2} < T < \infty$ and from the above (after a finite number of iterations) we see u_m^p is uniformly bounded (in m) in $L^T(\Omega)$ and we can apply the result from [5] (see Theorem A above) to see that u_m is bounded in L^∞.

Now recall from Proposition 1 there is some $C_1 > 0$ such that $\sup_\Omega u_m \geq C_1$ for all m. By a diagonal argument we can now find a subsequence u_{m_k} and some $u \in H_0^1 \cap L^\infty$ such that $u_{m_k} \rightharpoonup u$ in H_0^1 and $u_{m_k} \to u$ in L^T for all $T < \infty$ (after using convergence in L^2 and interpolation).
By the proof of Proposition 3.1 in [5] we have

$$\int_\Omega (a(x) \cdot \nabla u_m(x))^2 dx \leq \frac{C}{\lambda_m},$$

and hence u_m is approximating a first integral.
We now show that u is nonzero. From $(Q)_\lambda$ we have $\int_\Omega |\nabla u_m|^2 dx = \int_\Omega u_m^{p+1} dx$. By the critical Sobolev imbedding and using $2 < p+1 < 2^*$ we then get

$$C_N \left(\int_\Omega u_m^{2^*} dx \right)^{\frac{2}{2^*}} \leq \int_\Omega u_m^{p+1} dx \leq \left(\int_\Omega u_m^{2^*} dx \right)^{\frac{p+1}{2^*}} \left(\int_\Omega dx \right)^{1-\frac{p+1}{2^*}},$$

that implies

$$C_{N,\Omega} \leq \|u_m\|_{L^{2^*}}.$$

But recall that u_m is convergent (or at least some subsequence is) in L^T for all $T < \infty$. From this we see that we must have $C_{N,\Omega} \leq \|u\|_{L^{2^*}}$ and hence $u \neq 0$. $\square$

Proof of Theorem 2. We begin by recalling the Hardy-Sobolev inequality: let Ω denote a bounded smooth domain in $\mathbb{R}^N$ (with $N \geq 3$), $\tau \in [0,1]$ and q such that $\frac{1}{q} = \frac{1}{2^*} + \frac{\tau}{N}$. Then there is some $C = C(\Omega, \tau)$ such that

$$\|\frac{\zeta}{\delta^\tau}\|_{L^q} \leq C\|\nabla\zeta\|_{L^2}, \quad \textit{for all } \zeta \in H_0^1(\Omega),$$

where δ is the distance function as before. By changing C we can replace δ in the inequality with ψ, where ψ is promised since a satisfies (A2).
Now let u denote a positive smooth solution of $(Q)_\lambda$ with $\lambda > 1$. The following chain of inequalities will give us an estimate on the solution. At one we will use the Hardy-Sobolev inequality with $\tau := \frac{1}{N+1}$ and $q := \frac{2(N+1)}{N-1}$. By using the explicit value of q and τ we have

$$\begin{aligned}
\int_\Omega u^{2+\frac{2p}{N+1}} dx &= \int_\Omega u^{2+p(1-\frac{2}{q})} dx \\
&= \int_\Omega \frac{u^2}{\delta^{2\tau}} (u^p \delta)^{1-\frac{2}{q}} dx \\
&\leq \left(\int_\Omega \left\{ \frac{u^2}{\delta^{2\tau}} \right\}^{\frac{q}{2}} dx \right)^{\frac{2}{q}} \left(\int_\Omega (u^p\delta)^{(1-\frac{2}{q})\frac{q}{q-2}} dx \right)^{\frac{q-2}{q}} \\
&= \left(\int_\Omega \frac{u^q}{\delta^{\tau q}} dx \right)^{\frac{2}{q}} \left(\int_\Omega u^p \delta dx \right)^{\frac{q-2}{q}} \\
&\leq \|\frac{u}{\delta^\tau}\|_{L^q}^2 C \\
&\leq C_2 \int_\Omega |\nabla u|^2 dx \quad \text{by Hardy-Sobolev inequality} \\
&= C_2 \int_\Omega u^{p+1} dx \quad \text{from } (Q)_\lambda.
\end{aligned}$$

Note this gives us an estimate provided $2 + \frac{2p}{N+1} > p + 1$ and note this holds exactly when $p < \frac{N+1}{N-1}$. Hence for this range of p we have

$$\begin{aligned}
\int_\Omega u^{p+1} dx &\geq C_3 \int_\Omega u^{2+\frac{2p}{N+1}} dx \\
&= C_3 \int_\Omega (u^{p+1})^\alpha dx \qquad \text{where } \alpha := \frac{2 + \frac{2p}{N+1}}{p+1} > 1 \\
&\geq C\Big(\int_\Omega u^{p+1} dx \Big)^\alpha,
\end{aligned}$$

by Jensen's inequality. From this we see we get an L^{p+1} and H_0^1 estimate on u independent of λ and one can now perform an iteration as in the case of (A1) to see u is bounded in L^∞ independently of λ. We now proceed as in the case of (A1) to pass to a limit and to show the limit is nonzero. $\square$

3 Appendix

Sketch of a proof of Theorem A. We will use a variant of the De Giorgi method to sketch a proof. This variant uses an ODE approach but it can be proven using a De Giorgi iteration or a Moser iteration. Let $p > \frac{N}{2}$ and we assume N is large for simplicity of dealing with the exponents. Let $f \in L^p(\Omega)$ be smooth and we assume (for simplicity) that $f \geq 0$ in Ω. Take $\|f\|_{L^p} = 1$ and let u denote a nonnegative solution of $-\Delta u + b(x) \cdot \nabla u = f$ in Ω with $u = 0$ on $\partial\Omega$ with b smooth and $div(b) \leq 0$ in Ω. Let $\gamma = p'$ and hence we have $1 < \gamma < \gamma + 1 < q$ where $q = 2^*$. For $t \geq 0$ set

$$g(t) = \frac{1}{\gamma + 1} \int_\Omega (u(x) - t)_+^{\gamma+1} dx,$$

where + refers to the positive part of the function. Then note

$$-g'(t) = \int_\Omega (u - t)_+^\gamma dx.$$

By L^p interopolation we have

$$\|(u - t)_+\|_{L^{\gamma+1}} \leq \|(u - t)_+\|_{L^\gamma}^\theta \|(u - t)_+\|_{L^q}^{1-\theta},$$

where

$$\frac{1}{\gamma + 1} = \frac{\theta}{\gamma} + \frac{1 - \theta}{q}.$$

Rewriting this in terms of g and g' gives

$$(g(t))^{\frac{1}{\gamma+1}} \leq C(-g'(t))^{\frac{\theta}{\gamma}} \|(u - t)_+\|_{L^q}^{1-\theta},$$

where $C = C(p)$. We now use the equation to get the final term involving the L^q norm. Multiplying the equation by $(u-t)_+$ and integrating and using the fact $div(b) \leq 0$ we arrive at

$$\begin{aligned} C\|(u-t)_+\|_{L^q}^2 &\leq \int_\Omega |\nabla(u-t)_+|^2 dx \\ &\leq \|f\|_{L^p}\|(u-t)_+\|_{L^\gamma} \\ &= \|(u-t)_+\|_{L^\gamma} \\ &= (-g'(t))^{\frac{1}{\gamma}} \end{aligned} \tag{8}$$

where C is from the critical Sobolev imbedding. Subbing this into the previous inequality gives

$$(g(t))^{\frac{1}{\gamma+1}} \leq C(-g'(t))^{\frac{\theta}{\gamma}+\frac{1-\theta}{2\gamma}},$$

and this can be written as

$$g'(t) \leq -Cg(t)^\alpha,$$

where $\alpha = \frac{2\gamma}{(\gamma+1)(\theta+1)}$ and note the C only depends on p and N. Since $p > \frac{N}{2}$ (and recall we ar taking N big) this implies that $\gamma < \frac{N}{N-2}$ and after a computation one can see that $\alpha \in (0,1)$. Note that if $g(t) = 0$ for some $t > 0$ then we have $u \leq t$ a.e. in Ω. Lets assume this differential inequality is satisfied on $(0,T)$. Then we have

$$g(t)^{1-\alpha} \leq g(0)^{1-\alpha} - C(1-\alpha)t, \quad \forall 0 < t < T$$

and this shows that g decays to zero in finite time and also gives us the estimate that

$$T \leq \frac{g(0)^{1-\alpha}}{C(1-\alpha)},$$

note this decay to zero relies on the fact the exponent is strictly less than 1. So provided we can estimate $g(0)$ independent of f this gives us a uniform L^∞ on u as desired. But note that

$$g(0)(\gamma+1) = \int_\Omega u(x)^{\gamma+1} dx,$$

and from (8) we have

$$C\|(u-t)_+\|_{L^q}^2 \leq \|(u-t)_+\|_{L^\gamma},$$

which completes the proof after noting that $\gamma + 1 < q$.

□

Data Sets. Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

References

[1] A. Aghajani, C. Cowan, S.H. Lui, *Singular solutions of elliptic equations involving nonlinear gradient terms on perturbations of the ball*, J. Differential Equations 264, 4 (2018) 2865-2896.

[2] A. Ambrosetti, A. Malchiodi, **Nonlinear analysis and semilinear elliptic problems**, Cambridge University Press; 2007 Jan 4.

[3] H. Berestycki, F. Hamel and N. Nadirashvili, *The principal eigenvalue of elliptic operators with large drift and applications to nonlinear propagation phenomena*, Comm. Math. Phys. 253 (2005), pp. 451-480.

[4] H. Berestycki, F. Hamel and N. Nadirashvili, *The speed of propagation for KPP type problems. I - Periodic framework*, J. Eur. Math. Soc. 7 (2005), pp. 173-213.

[5] H. Berestycki, A. Kiselev, A. Novikov and L. Ryzhik, *The explosion problem in a flow*, JAMA, 110, 2010, 31-65.

[6] H. Brezis, R.E.L. Turner, *On a class of superlinear elliptic problems*, Comm. Partial Differential Equations 2 (1977) 601-614.

[7] Z. Chen, C.-S. Lin, W. Zou, *Monotonicity and nonexistence results to cooperative systems in the half space*, J. Funct. Anal. 266 (2014), 1088-1105.

[8] J.M. Coron, *Topologie et cas limite des injections de Sobolev.* C.R. Acad. Sc. Paris, 299, Series I, 209–212.(1984).

[9] C. Cowan, *Optimal Hardy inequalities for general elliptic operators with improvements*, Communications on Pure & Applied Analysis, 109-140, (2009).

[10] E.N. Dancer, *Some notes on the method of moving planes*, Bull. Austral. Math. Soc. 46 (3) (1992) 425-434.

[11] E.N. Dancer, *Stable solutions on $\mathbb{R}^n$ and the primary branch of some non-self-adjoint convex problems*, Differential and Integral Equations 17(2004), 961-970.

[12] E.N. Dancer and A. Farina, *On the classification of solutions of $-\Delta u = e^u$ on $\mathbb{R}^N$: stability outside a compact set and applications,* Proc. Amer. Math. Soc. 137 (2009),1333-1338.

[13] D.G. de Figueiredo, P.-L. Lions, R.D. Nussbaum, *A priori estimates and existence of positive solutions of semilinear elliptic equations*, J. Math. Pures Appl. 61 (1982) 41-63.

[14] M. del Pino, J. Dolbeault and M. Musso, *A phase plane analysis of the multi-bubbling phenomenon in some slightly supercritical equations*, Monatsh. Math. 142 no. 1-2, 57-79 (2004).

[15] M. del Pino, J. Dolbeault and M. Musso, *Bubble-tower radial solutions in the slightly supercritical Brezis-Nirenberg problem*, Journal of Differential Equations 193 (2003), no. 2, 280-306.

[16] M. del Pino, P. Felmer, M. Musso, Multi-bubble solutions for slightly super-critical elliptic problems in domains with symmetries Bull. London Math. Society 35 (2003), no. 4, 513-521

[17] M. del Pino, P. Felmer and M. Musso, *Two-bubble solutions in the super-critical Bahri-Coron's problem.* Calc. Var. Partial Differential Equations 16(2):113–145.(2003).

[18] M. del Pino and J. Wei, *Supercritical elliptic problems in domains with small holes,* Ann. Non linearie, Annoles de l'Institut H. Poincare, 24(2007), no.4, 507-520.

[19] L. Dupaigne, **Stable solutions of elliptic partial differential equations**, Chapman and Hall/CRC Monographs and surveys in Pure and applied mathematics 143, (2011).

[20] L. Dupaigne, B. Sirakov and P. Souplet, *A Liouville-Type Theorem for the Lane–Emden Equation in a Half-space*, International Mathematics Research Notices (2021).

[21] A. Farina, *On the classification of solutions of the Lane–Emden equation on unbounded domains of R^N*, J. Math. Pures Appl. 87 (2007) 537-561.

[22] B. Gidas and J. Spruck, *A priori bounds for positive solutions of non-linear elliptic equations*, Comm. Partial Differential Equations 6 (1981), 883-901.

[23] B. Gidas and J. Spruck *Global and local behavior of positive solutions of nonlinear elliptic equations*, Comm. Pure Appl. Math. 34 (1981), 525-598.

[24] S. Kirsch and M. El Smaily, *The speed of propagation for KPP reaction-diffusion equations within large drift*, Advances in Differential Equations, 16, Numbers 3-4, (2011), pp. 361-400.

[25] S. Kirsch and M. El Smaily, *Asymptotics of the KPP minimal speed within large drift*, Comptes Rendus de l'Académie des Sciences, 348, Issues 15-16, August 2010, pp. 857-861.

[26] S. Kirsch and M. El Smaily, *Front speed enhancement by incompressible flows in three or higher dimensions*, Arch. Ration. Mech. Anal., 213, Issue 1 (2014), pp. 327-354.

[27] J. McGough, J. Mortensen, *Pohozaev obstructions on non-starlike domains.* Calc. Var. Partial Differential Equations 18 (2003), no. 2, 189–205.

[28] J. McGough, J. Mortensen, C. Rickett and G. Stubbendieck, *Domain geometry and the Pohozaev identity.* Electron. J. Differential Equations 2005, No. 32, 16 pp.

[29] J. Mawhin, *Leray-Schauder degree: a half century of extensions and applications*, Topological methods in nonlinear analysis. 1999;14(2):195-228.

[30] D. Passaseo, *Nonexistence results for elliptic problems with supercritical nonlinearity in nontrivial domains.* J. Funct. Anal. 114(1):97-105.(1993).

[31] S. Pohozaev, *Eigenfunctions of the equation* $-\Delta u + \lambda f(u) = 0$*,* Soviet Math. Doklady 6 (1965), 1408-1411.

[32] P. Poláčik, P. Quittner and P. Souplet, *Singularity and decay estimates in superlinear problems via Liouville-type theorems, I: Elliptic equations and systems*, Duke Math. J. Volume 139, Number 3 (2007), 555-579.

[33] [16] P. Quittner, Ph. Souplet, *A priori estimates and existence for elliptic systems via bootstrap in weighted Lebesgue spaces*, Arch. Ration. Mech. Anal. 174 (2004) 49-81.

[34] W. Reichel and T. Weth, *Existence of solutions to nonlinear subcritical higher order elliptic Dirichlet problems*, J. Differential Equations 248 (2010), 1866-1878

[35] L. Ryzhik and A. Zlatos, KPP pulsating front speed-up by flows, Comm. Math. Sci. 5 (2007), pp. 575-593

[36] R. Schaaf, *Uniqueness for semilinear elliptic problems: supercritical growth and domain geometry*, Adv. Differential Equations, 5:10–12 (2000), pp. 1201–1220.

[37] K. Schmitt, *Positive solutions of semilinear elliptic boundary value problems*, Topological methods in differential equations and inclusions (Montreal, PQ, 1994), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 472, Kluwer Academic Publishers, Dordrecht, 1995, pp. 447-500.

[38] P. Souplet, *Optimal regularity conditions for elliptic problems via L^p_δ -spaces*, Duke Math. J. 127 (2005) 175-192.

[39] M. Struwe, (1990). Variational Methods – Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems. Berlin: Springer-Verlag.

[40] A. Zlatos, A.: Sharp asymptotics for KPP pulsating front speed-up and diffusion enhancement by flows. Arch. Ration. Mech. Anal. 195(2), 441-453 (2010)